The \( \ell_p \)-norm is a convex surrogate of the \( \ell_0 \)-norm, but it may be too loose. A better choice is to use nonconvex surrogate functions, e.g., \( \ell_p \)-norm (0 < p < 1).

The nuclear norm is a convex surrogate of the rank function, but it may be too loose. A better choice is to apply the nonconvex surrogate functions of \( \ell_p \)-norm on the singular values.

However, the nonconvex low-rank minimization is much more challenging than the nonconvex sparse minimization. We propose a general solver for nonconvex low-rank minimization.

Many Nonconvex Surrogate Functions of the \( \ell_p \)-norm

<table>
<thead>
<tr>
<th>Functions</th>
<th>( \ell_p )-norm</th>
<th>Supergradient ( \partial g(x, \lambda \theta) )</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>SCAD</td>
<td>( \max {0,</td>
<td>x</td>
<td>- \lambda } )</td>
</tr>
<tr>
<td>MCP</td>
<td>( \frac{\lambda \theta}{</td>
<td>x</td>
<td>+ \lambda \theta} \cdot</td>
</tr>
<tr>
<td>Logarithm</td>
<td>( \frac{1}{\mu} \log_+ (1 + \frac{x}{\mu}) )</td>
<td>( \frac{1}{\mu} \cdot 1_{{</td>
<td>x</td>
</tr>
<tr>
<td>ETP</td>
<td>( \frac{1}{\lambda} \left(</td>
<td>x</td>
<td>^{1 - \lambda} - 1 \right) )</td>
</tr>
</tbody>
</table>

The same properties:
- All the above nonconvex surrogate functions are concave and monotonically increasing on \([0, \infty)\).
- Their supergradients are nonnegative and monotonically decreasing.

**Convergence Analysis of IRNN**

Theorem A2. The sequence \( \{x^k\} \) generated by IRNN satisfies the following properties:

1. \( F(x^k) - F(x^* + 1) \) is monotonically decreasing.
2. If \( x^0 \) is a stationary point of \( F + \Omega \), then \( x^k \) is a stationary point of \( F + \Omega + \Psi \).
3. If \( x^0 \) is a stationary point of \( F + \Omega \), then \( x^k \) is a stationary point of \( F + \Omega + \Psi \).
4. If \( x^0 \) is a stationary point of \( F + \Omega \), then \( x^k \) is a stationary point of \( F + \Omega + \Psi \).

**Take Home Message**

- Use nonconvex surrogate rank function instead of nuclear norm.
- Iteratively Reweighted Nuclear Norm (IRNN), the first general solver to (2), was proposed.

Test on the following problem with different nonconvex surrogate functions:

\[
\min_{X \in \mathbb{R}^{m \times n}} F(X) = \frac{1}{2} \| X - Y \|_F^2 + \frac{c}{2} \lambda \sigma(X),
\]

where \( \Omega \) is the index set, and \( \lambda : \mathbb{R}^{m \times n} \to \mathbb{R}^{m \times n} \) is a linear operator that keeps the entries in \( \Omega \) unchanged and those outside \( \Omega \) zeros.

**Experiment: Low-rank Matrix Completion on Random Data**

- Original Image
- Noisy Image
- IRNN-Lp, IRNN-SCAD, IRNN-Logarithm, IRNN-MCP

**Experiment: Low-rank Matrix Completion for Image Recovery**

- Original Image
- Noisy Image
- IRNN-Lp, IRNN-SCAD, IRNN-Logarithm, IRNN-MCP

**Figure:** Comparison of the PSNR values by different matrix completion algorithms.